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INTIAL BOUNDARY VALUE PROBLEM FOR THE LINEARIZED KDV EQUATION ON STAR GRAPH WITH ONE SEMI-INFINITE AND TWO BOUNDED BONDS

Akhmedov Maksad
Kadirov Rafik
YEOJU TECHNICAL INSTITUTE IN TASHKENT

Annotation. In this article, we investigated an initial boundary value problem for the linearized KDV equation on a simple metric graph consisting of two bounded segments and one semi-infinite straight line connected at one point, called the vertex of the graph. The uniqueness of the solution is proved by the method of energy integrals.

Using the potential method, an integral formula is constructed for solving the problem under consideration.

Key words: Third order PDE, boundary value problem, method of energy integrals, method of potentials, initial condition, boundary condition, integral equation.

Аннотация.В данной статье мы исследовали начально - краевую задачу для линеаризованного уравнения КДВ на простом метрическом графе, состоящем из двух ограниченных отрезков и одной полу-бесконечной прямой, соединенных в одной точке, называемой вершиной графа. Единственность решения доказана методом интегралов энергии.

С помощью метода потенциалов построена интегральная формула для решения рассматриваемой задачи.

Annotatsiya. Mazkur maqolada ikkita chekli kesma va bitta cheksiz grafning uchi deb ataluvchi bitta nuqtada birlashtirishdan hosil bo'lgan sodda metrik grafda chiziqli KDV tenglamasi uchun boshlang'ich chegaraviy masala qaralgan. Masala yechimi yagonaligi energiya integrallari usulida isbotlangan. Potensiallar usulidan foydalanib masala yechimi integral formula olingan.

1. INTRODUCTION

The Korteweg-de Vries (KdV) equation has attracted attention of both physical scientists and mathematicians, since it was found to admit soliton solutions and be able to model the propagation of solitary wave on the water surface, aphenomena first discovered by Scott Russellin1834.Theequationisalso used, e.g., to model the unidirectional propagation of small amplitude long waves in nonlinear dispersive systems such as the ion-acoustic waves in a collisionless plasma, and the magnetosonic waves in a magnetized plasma etc [11]. The linearized KdV provides an asymptotic description of linear, undirectional, weakly dispersive long waves, for example, shallow water waves. In [12] it is proven that via normal form transforms the solution of the KdV equation can be reduced to the solution of the linear KdV equation. Belashov and Vladimirov [12] numerically investigate evolution of the single disturbance $u(0,x)=u_0\exp(-x^2/l^2)$ and show that in the limit $l\to 0$, $u_0 l^2 = const$, the solution of the KdV equation is qualitatively similar to the solution of linearized KdV equation. Boundary value problems on half lines are considered in [2,5,7].

In this paper, we address the linearized KdV equation on a star graph Γ with two bounded and one semi-infinite bonds connected at one point, called the vertex. The bonds are denoted by B_j, $j=1,2,3,$ the coordinate x_1 on B_1 is defined from $-L_1$ to 0, and coordinate x_2 on B_2 is defined from 0 to L_2, and coordinate x_3 on B_3 is defined from 0 to $+\infty$. On each bond we consider the linear equation:

$$\left(\frac{\partial}{\partial t}-\frac{\partial^3}{\partial_j^3}\right)u_j(x_j,t)=f_j(x,t),\quad t>0, x_j\in B_j, j=1,2,3. \tag{1}$$

Formulation of the problems For the above star graph, we need to impose 5 BCs at the vertex point, which should also connection between the bonds and 2 BCs at the right side of B_2 and B_3. In detail, we require: B_2 and B_3. In detail, we require:

$$u_1(0,t)=a_2u_2(0,t)=a_3u_3(0,t),\ u_{1x}(0,t)=b_2u_{2x}(0,t)=b_3u_{3x}(0,t), \tag{2}$$

$$u_{1xx}(0,t)=\frac{1}{a_2}u_{2xx}(0,t)+\frac{1}{a_3}u_{3xx}(0,t), \tag{3}$$

$$u_1(-L_1,t)=\phi_1(t),\ u_2(\mathrm{L}_2,t)=\phi_2(t),\ u_{2x}(\mathrm{L}_2,t)=\delta_2(t), \tag{4}$$

for $0<t<T,\quad T=const.$

Furthermore, we assume that the functions $f_j(x,t),\ j=1,2,3,$ are smooth enough and bounded. The initial conditions are given by:

$$u_j(x,0)=0, x\in\overline{B_j},\ j=1,2,3. \tag{5}$$

It should be noted that the above vertex conditions are not the only possible ones. The main motivation for our choice is caused by the fact that they guarantee uniqueness of the solution and, if the solutions decay (to zero) at infinity, the norm (energy) conservation.

2. Existence and uniqueness of solutions

Lemma 1. Let $\frac{1}{b_2^2}+\frac{1}{b_3^2}\le 1$. Then the problem (1)-(5) has at most one solution.

$$\frac{d}{dt}\int_a^b u_j^2(x,t)dx=(2u_ju_{jxx}-u_{jx}^2)|_a^b+2\int_a^b f_j(x,t)u_j(x,t)dx$$

for appropriate values of constants a and b on each bond. We put $\phi_0(t)\equiv 0$. Then, the above equalities and vertex conditions (2)-(5) yield

$$\frac{d}{dt}\left(e^{-\varepsilon t}\|u\|^2\right) \le e^{-\varepsilon t}\left(\frac{1}{\varepsilon^2}\|f\|^2 + \phi_1^2(t)\right)$$

$$\|u\|^2 \le \int_0^t e^{-\varepsilon(t-\tau)}\left(\frac{1}{\varepsilon^2}\|f(\cdot,\tau)\|^2 + \delta_2^2(\tau)\right)d\tau, \qquad (6)$$

where

$$\left(u,v\right) = \int_{-L_1}^{0} u_1 v_1 dx_1 + \int_0^{L_2} u_2 v_2 dx_2 + \int_0^{+\infty} u_3 v_3 dx_3,$$

$\|u\| = \sqrt{(u,u)}$, are scalar product and norm defined on graph ε is an arbitrary positive number. Uniqueness of the solution follows from (6).

Theorem 1. Let, $\frac{1}{b_2^2} + \frac{1}{b_3^2} \le 1,\ \phi_0(t) \in C^2[0,\ T],\ \phi_1(t) \in C^1[0,\ T].$

Then the problem (1)-(5) has a unique solution in $C^1([0,\ T],\ C^3(\Gamma))$.

Proof of theorem. To prove the theorem, we use the following functions are called fundamental solutions of the equation $u_t - u_{xxx} = 0$.

$$U\left(x,t;\xi,\eta\right) = \begin{cases} \frac{1}{(t-\eta)^{\frac{1}{3}}} f\left(\frac{x-\xi}{(t-\eta)^{\frac{1}{3}}}\right), & t > \eta, \\ 0 & t \le \eta \end{cases}$$

$$V\left(x,t;\xi,\eta\right) = \begin{cases} \frac{1}{(t-\eta)^{\frac{1}{3}}} \varphi\left(\frac{x-\xi}{(t-\eta)^{\frac{1}{3}}}\right), & t > \eta, \\ 0 & t \le \eta \end{cases}$$

where $f(x) = \frac{\pi}{\sqrt[3]{3}} Ai\left(-\frac{x}{\sqrt[3]{3}}\right),\ \varphi(x) = \frac{\pi}{\sqrt[3]{3}} Bi\left(-\frac{x}{\sqrt[3]{3}}\right)$ for $x \ge 0,\ \varphi(x) = 0$ for $x < 0$ and $Ai(x)$ and $Bi(x)$ are the Airy functions. The functions $f(x)$ and $\varphi(x)$ are integrable and $\int_{-\infty}^{0} f(x)dx = \frac{\pi}{3},\ \int_0^{+\infty} f(x)dx = \frac{2\pi}{3},\ \int_0^{+\infty}\varphi(x)dx = 0.$ Below, we also use fractional integrals[8]:

$$J_{(0,t)}^{\alpha} f(t) := \frac{1}{\Gamma(\alpha)}\int_0^t \left(t-\tau\right)^{\alpha-1} f\left(\tau\right)d\tau,\ 0 < \alpha < 1$$

and the inverse of this operator, i.e., the Riemann-Liouville fractional derivatives [8,9] defined by:

$$D_{(0,t)}^{\alpha} f(t) := \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t \left(t-\tau\right)^{-\alpha} f\left(\tau\right)d\tau,\ 0 < \alpha < 1.\$$$

We look for solution in the form:

$$u_1(x,t) = \int_0^t U(x,t;0,\eta)\varphi_1(\eta)d\eta + \int_0^t V(x,t;0,\eta)\psi_1(\eta)d\eta +$$

$$+\int_0^t U(x,t;-L_1,\eta)\alpha_1(\eta)d\eta + F_1(x,t)$$

$$u_2(x,t) = \int_0^t U(x,t;0,\eta)\varphi_2(\eta)d\eta + \int_0^t V(x,t;L_2,\eta)\alpha_2(\eta)d\eta +$$

$$+\int_0^t U(x,t;L_2,\eta)\beta_2(\eta)d\eta + F_2(x,t)$$

$$u_3(x,t) = \int_0^t U(x,t;0,\eta)\varphi_3(\eta)d\eta + \int_0^t V(x,t;0,\eta)\alpha_3(\eta)d\eta + F_3(x,t)$$

where

$$F_k(x,t) = \frac{1}{\pi}\int_0^t \int_{B_k} U(x,t;\xi,\eta) f_k(\xi,\eta)d\xi d\eta,\ k=1,2,3. \tag{7}$$

Satisfying the conditions (2) we have:

$$\int_0^t \frac{1}{(t-\eta)^{\frac{1}{3}}}(f(0)\varphi_1(\eta) + \varphi(0)\psi_1(\eta) + f\left(\frac{L_1}{(t-\eta)^{\frac{1}{3}}}\right)\alpha_1(\eta) - a_2 f(0)\varphi_2(\eta) -$$

$$-a_2\varphi\left(0\right)\alpha_2(\eta) - f\left(-\frac{L_2}{(t-\eta)^{\frac{1}{3}}}\right)\beta_2(\eta))d\eta = a_2F_2(0,t) - F_1(0,t).$$

From here, according to properties of fractional derivatives, one can get

$$f(0)\varphi_1(t) + \varphi(0)\psi_1(t) - a_2 f(0)\varphi_2(t) - a_2\varphi\left(0\right)\alpha_2(\eta) +$$

$$+\frac{1}{\Gamma\left(\frac{1}{3}\right)} D_{(0,t)}^{\frac{2}{3}} \int_0^t \frac{1}{(t-\eta)^{\frac{1}{3}}}\left[f\left(\frac{L_1}{(t-\eta)^{\frac{1}{3}}}\right)\alpha_1(\eta) - a_2 f\left(-\frac{L_2}{(t-\eta)^{\frac{1}{3}}}\right)\beta_2(\eta)\right] d\eta =$$

$$= \frac{1}{\Gamma\left(\frac{1}{3}\right)} D_{(0,t)}^{\frac{2}{3}} [a_2F_2(0,t) - F_1(0,t)], \tag{8}$$

Analogously, from the second part of this condition we get

$$f(0)\varphi_1(t) + \varphi(0)\psi_1(t) - a_3 f(0)\varphi_3(t) - a_3\varphi(0)\alpha_3(t) +$$

$$+\frac{1}{\Gamma\left(\frac{1}{3}\right)} D_{(0,t)}^{\frac{2}{3}} \int_0^t \frac{1}{(t-\eta)^{\frac{1}{3}}}(f\left(\frac{L_1}{(t-\eta)^{\frac{1}{3}}}\right)\alpha_1(\eta) =$$

$$= \frac{1}{\Gamma\left(\frac{1}{3}\right)} D_{(0,t)}^{\frac{2}{3}} \left[a_3F_3(0,t) - F_1(0,t)\right], \tag{9}$$

$$f'(0)\varphi_1(t) + \varphi'(0)\psi_1(t) - b_2 f'(0)\varphi_2(t) - b_2\varphi'(0)\alpha_2(t) +$$

$$+\frac{1}{\Gamma\left(\frac{2}{3}\right)}D_{(0,t)}^{\frac{1}{3}}\int_0^t \frac{1}{(t-\eta)^{\frac{2}{3}}} f'\left(\frac{L_1}{(t-\eta)^{\frac{1}{3}}}\right)\alpha_1(\eta)d\eta -$$

$$-b_2\frac{1}{\Gamma\left(\frac{2}{3}\right)}D_{(0,t)}^{\frac{1}{3}}\int_0^t \frac{1}{(t-\eta)^{\frac{2}{3}}} f'\left(-\frac{L_2}{(t-\eta)^{\frac{1}{3}}}\right)\beta_2(\eta)d\eta -$$

$$=\frac{1}{\Gamma\left(\frac{2}{3}\right)}D_{(0,t)}^{\frac{1}{3}}\left[b_2F_{2x}(0,t)-F_{1x}(0,t)\right], \qquad (10)$$

$$f'(0)\varphi_1(t)+\varphi'(0)\psi_1(t)-b_3f'(0)\varphi_3(t)-b_3\varphi'(0)\alpha_3(t)+$$

$$+\frac{1}{\Gamma\left(\frac{2}{3}\right)}D_{(0,t)}^{\frac{1}{3}}\int_0^t \frac{1}{(t-\eta)^{\frac{2}{3}}} f'\left(\frac{L_1}{(t-\eta)^{\frac{1}{3}}}\right)\alpha_1(\eta)d\eta =$$

$$=\frac{1}{\Gamma\left(\frac{2}{3}\right)}D_{(0,t)}^{\frac{1}{3}}\left[b_3F_{3x}(0,t)-F_{1x}(0,t)\right], \qquad (11)$$

According to the vertex condition $u_{1xx}(0,t)=\frac{1}{a_2}u_{2xx}(0,t)+\frac{1}{a_3}u_{3xx}(0,t)$

we obtain

$$-\frac{2\pi}{3}\cdot\varphi_1(t)+\frac{1}{a_2}\cdot\frac{2\pi}{3}\cdot\varphi_2(t)+\frac{1}{a_3}\cdot\frac{2\pi}{3}\cdot\varphi_3(t)+$$

$$+\int_0^t\frac{1}{t-\eta}\left[f''\left(\frac{L_1}{(t-\eta)^{\frac{1}{3}}}\right)\alpha_1(\eta)-\frac{1}{a_2}\cdot f''\left(-\frac{L_2}{(t-\eta)^{\frac{1}{3}}}\right)\beta_2(\eta)=\right.$$

$$=\frac{1}{a_3}F_{3xx}(0,t)+\frac{1}{a_2}F_{2xx}(0,t)-F_{1xx}(0,t),$$

(12)

From the boundary conditions (4) we get

$$f(0)\alpha_1(t)+\frac{1}{\Gamma\left(\frac{1}{3}\right)}D_{(0,t)}^{\frac{2}{3}}\int_0^t \frac{1}{(t-\eta)^{\frac{1}{3}}} f\left(\frac{-L_1}{(t-\eta)^{\frac{1}{3}}}\right)\varphi_1(\eta)d\eta +$$

$$+\frac{1}{\Gamma\left(\frac{1}{3}\right)}D_{(0,t)}^{\frac{2}{3}}\int_0^t \frac{1}{(t-\eta)^{\frac{1}{3}}} \varphi\left(\frac{-L_1}{(t-\eta)^{\frac{1}{3}}}\right)\psi_1(\eta)d\eta$$

$$=\frac{1}{\Gamma\left(\frac{1}{3}\right)} D_{(0,t)}^{\frac{2}{3}}\left[\phi_1(t)-F_1(-\mathrm{L}_1,t)\right], \tag{13}$$

$$f(0)\varphi_2(t)+\varphi(0)\alpha_2(t)+\frac{1}{\Gamma\left(\frac{1}{3}\right)} D_{(0,t)}^{\frac{2}{3}}\int_0^t \frac{1}{(t-\eta)^{\frac{1}{3}}} f\left(\frac{L_2}{(t-\eta)^{\frac{1}{3}}}\right)\beta_2(\eta)d\eta=$$

$$=\frac{1}{\Gamma\left(\frac{1}{3}\right)} D_{(0,t)}^{\frac{2}{3}}\left[\phi_2(t)-F_2(\mathrm{L}_2,t)\right], \tag{14}$$

$$f'(0)\varphi_2(t)+\varphi'(0)\alpha_2(t)+\frac{1}{\Gamma\left(\frac{2}{3}\right)} D_{(0,t)}^{\frac{1}{3}}\int_0^t \frac{1}{(t-\eta)^{\frac{2}{3}}} f'\left(\frac{L_2}{(t-\eta)^{\frac{1}{3}}}\right)\beta_2(\eta)d\eta=$$

$$=\frac{1}{\Gamma\left(\frac{2}{3}\right)} D_{(0,t)}^{\frac{1}{3}}\left[\delta_1(t)-F_{1x}(-\mathrm{L}_1,t)\right], \tag{15}$$

We obtained the system of integral equations (8)-(15) with respect to unknowns $\Phi(t)=(\varphi_1(t),\varphi_2(t),\varphi_3(t),\psi_1(t),\alpha_1(t),\alpha_2(t),\alpha_3(t),\beta_2(t))^T$.

The matrix A

$$A=\begin{pmatrix} f(0) & -a_2 f(0) & 0 & 0 & 0 & 0 & 0 & 0 \\ f(0) & 0 & -a_3 f(0) & 0 & 0 & 0 & -a_3\varphi(0) & 0 \\ f'(0) & -b_2 f'(0) & -b_3 f'(0) & \varphi'(0) & 0 & -b_2\varphi'(0) & -b_2\varphi'(0) & 0 \\ -2\pi/3 & 2\pi/3a_2 & 2\pi/3a_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \varphi(0) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & f(0) \\ 0 & f(0) & 0 & 0 & 0 & \varphi(0) & 0 & 0 \\ 0 & f'(0) & 0 & 0 & 0 & \varphi'(0) & 0 & 0 \end{pmatrix}$$

$$\det A=\frac{-2\pi^3}{9}\cdot\frac{1}{\Gamma\left(\frac{1}{3}\right)}$$

Under conditions of the theorem this determinant is not singular.

According to the asymptotes of Airy functions the kernels of the integral operators are integrable. Hence, it follows from the uniqueness theorem and Fredholm alternatives that the system of equations has a unique solution. In this way the solvability of the problem is proved.

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