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COMPARATIVE ANALYSIS OF EXPONENTIAL AND SUB-EXPONENTIAL INTEGER FACTORIZATION ALGORITHMS

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Abstract. Integer factorization remains one of the central challenges in modern number theory and continues to play a decisive role in the security of contemporary cryptographic systems. Asymmetric cryptographic schemes—especially RSA—derive their strength from the practical difficulty of breaking down a large composite number into its prime factors. With continuous progress in both computing power and algorithmic design, understanding how well different factorization methods perform has become increasingly important for anticipating cryptographic resilience and identifying where vulnerabilities might emerge. This paper provides an in-depth comparison of the most prominent exponential and sub-exponential factorization algorithms. Among the methods examined are classical approaches such as Fermat's technique, Pollard's ρ, Pollard–Strassen, Shanks' SQUFOF, Pollard's p−1, and Lehman's algorithm, as well as more advanced strategies including Dixon's method, the Continued Fraction Method (CFRAC), the Quadratic Sieve, Lenstra's Elliptic Curve Method, and the Number Field Sieve. Each algorithm is explored through its mathematical foundation, computational complexity, structural characteristics, and practical range of effectiveness. By bringing together theoretical perspectives and

established performance insights, the paper offers a coherent view of how factorization algorithms have evolved and why they matter for the security of modern cryptographic systems.

1. Introduction

Decomposing a composite number into its prime factors is a problem with a long intellectual history, yet it remains deeply relevant in today's digital era. While multiplying two large primes is a trivial task for any computer, reversing the process - recovering those primes from their product - is inherently difficult when the numbers are sufficiently large. This inherent imbalance forms the backbone of public-key cryptography. In particular, the security of RSA hinges on the assumption that factoring a number of the form $N=pq$, where p and q primes typically between 1024 and 4096 bits, is computationally infeasible with classical algorithms [1].

Over time, researchers have introduced a wide variety of factorization algorithms, each shaped by different mathematical ideas and optimized for different scenarios. Some of the earliest approaches, including Fermat's method and Pollard's ρ, are conceptually simple and rely on clever number-theoretic observations. However, their performance deteriorates rapidly as numbers grow larger. Sub-exponential algorithms, such as the Quadratic Sieve and the Number Field Sieve, brought a major shift by dramatically reducing the time required to factor large semiprimes. These algorithms form the basis for many of the most significant factorization achievements reported in recent decades.

Studying these methods closely is important for several reasons. First, it helps assess how secure widely used cryptographic systems truly are. Second, it provides insight into how an algorithm behaves under different numerical structures, such as when the primes are close to each other or when one of them has a smooth component. Third, as computing technologies evolve - including the rise of specialized hardware and large-scale distributed computation - regularly reviewing algorithmic capabilities

ensures that cryptographic recommendations remain aligned with real-world capabilities.

The analysis in this paper builds on the detailed explanations and examples provided in the accompanying source material [2], reorganizing them into a clear comparative framework. The aim is to offer researchers, students, and practitioners a well-structured and accessible overview of both classical and modern factorization techniques, emphasizing their strengths, limitations, and relevance in contemporary cryptography.

2. Methodology

To produce a balanced and reliable comparative analysis, this study adopts a multi-stage methodological approach consisting of classification, theoretical review, empirical interpretation, and structured comparison.

2.1 Algorithm Classification

The algorithms were first divided into two major families:

- Exponential-time algorithms, whose running times grow roughly as $2^{O(n)}$ or with similar exponential characteristics.
- Sub-exponential algorithms, typically described by the L-notation, which achieve considerably faster performance for large input sizes.

This classification aligns with established conventions in computational number theory and reflects the distinctions highlighted in the source document [2].

2.2 Analytical Framework

Each algorithm is examined along five primary dimensions:

- Mathematical foundation - e.g., difference of squares, cycle detection, elliptic curves, or algebraic number fields.
- Asymptotic time complexity - expressed in classical big-O notation or the L-notation for subexponential algorithms.
- Effective operational range - approximate bit-lengths where the algorithm remains practical based on documented performance [2-13].

- Strengths and weaknesses - stemming from structural behavior or dependence on specific properties such as smoothness.
- General-purpose vs. special-purpose use - determining whether the algorithm is broadly applicable or effective only when certain number-theoretic conditions are met.

2.3 Empirical Interpretation

The source material includes computed examples and observed execution times for different algorithms when applied to integers of varying sizes [2]. These examples were not replicated experimentally but were incorporated as *qualitative indicators* of practical performance. The interpretation focuses on identifying trends - such as the rapid degradation of exponential methods or the consistent scalability of sub-exponential algorithms.

2.4 Comparative Synthesis

The final stage involved constructing summary tables for both exponential and sub-exponential algorithms. These tables synthesize theoretical and practical insights into clear, academically formatted comparisons without altering the original ordering of referenced literature.

This methodology ensures that the analysis maintains academic rigor while remaining faithful to the content and structure of the provided material.

3. Exponential-Time Algorithms

3.1 Fermat's Factorization Method

Fermat's approach relies on representing

$$N = x^2 - y^2 = (x - y)(x + y)$$

It is efficient only when the prime factors p and q are close in magnitude. For unbalanced semiprimes its running time becomes exponential in the bit-length of N[3]. Effective for $<$ 16-bit inputs.

3.2 Pollard's ρ Algorithm

Pollard's ρ uses pseudorandom iterates and Floyd's cycle-finding technique. Its expected runtime is approximately

$$O(\sqrt{p}),$$

where p is the smallest prime factor. It performs well for medium-sized factors and is practical up to ~ 30 bits [4].

3.3 Pollard-Strassen Algorithm

This hybrid method combines Pollard's ρ with Strassen's enhancements, improving factor discovery in specific conditions but remaining exponential in the worst case [5]. Applicable roughly below 24 bits.

3.4 Shanks' SQUFOF (Square Forms Factorization)

SQUFOF refines Fermat's idea using quadratic forms. It is fast on small devices and performs well for numbers below ~ 26 bits [6].

3.5 Pollard's $p-1$ Algorithm

This method exploits the smoothness of $p-1$. When a factor satisfies the smoothness condition, the algorithm is extremely efficient; otherwise it fails entirely. It is effective for some integers up to ~ 66 bits [7].

3.6 Lehman's Algorithm

Lehman's algorithm is deterministic and achieves a complexity close to

$$O\left(N^{1/3}\right),$$

representing a historical improvement over naïve trial division. It is mainly of theoretical interest and practical only for small values (<16 bits) [8].

4. Sub-Exponential Factorization Algorithms

Sub-exponential algorithms are described using the L-notation:

$$L_n(\alpha, c) = \exp((c + o(1))(\log n)^{\alpha}(\log\log n)^{1-\alpha})$$

4.1 Dixon's Algorithm

One of the earliest sub-exponential methods, Dixon's algorithm finds random squares modulo N and uses them to derive congruences. Effective below ~ 73 bits and of foundational importance [9].

4.2 CFRAC (Continued Fraction Factorization Method)

CFRAC uses convergents in the continued fraction expansion of $\sqrt{N}$ to find congruent squares. It was one of the fastest known methods prior to the Quadratic Sieve and is practical up to ~ 80 bits [10].

4.3 Quadratic Sieve (QS)

The Quadratic Sieve generalizes Fermat's method with sieving and linear algebra, achieving complexity

$$L_n\left(\frac{1}{2}, 1\right).$$

It is highly effective for numbers up to ~ 100 digits (≈ 330 bits) and remains one of the most influential general-purpose algorithms [11].

4.4 Lenstra's Elliptic Curve Method (ECM)

ECM uses arithmetic on elliptic curves to find relatively small prime factors. Its runtime approximates

$$L_p\left(\frac{1}{2}, \sqrt{2}\right),$$

making it the most efficient method for discovering small and medium-sized prime factors (<83 bits) [12].

4.5 Number Field Sieve (NFS)

NFS is the fastest known classical factoring algorithm, with complexity

$$L_n\left(\frac{1}{3}, (64/9)^{1/3}\right)$$

It is the dominant method for factoring numbers larger than ~ 110 bits and the only practical approach for modern RSA key sizes [13].

Comparative Analysis

5.1 Exponential Algorithms Summary

Table 1. Summary of Exponential-Time Factorization Algorithms

Algorithm	**Mathematical Basis**	**Effective Range**	**Complexity**
Fermat	Difference of squares	<16 bits	Exponential [3]
Pollard ρ	Cycle detection	<30 bits	$O(\sqrt{p})$[4]
Pollard-Strassen	Hybrid	<24 bits	Exponential [5]
SQUFOF	Quadratic forms	<26 bits	Exponential [6]
Pollard $p-1$	Smoothness	<66 bits	Smoothness-dependent [7]
Lehman	Deterministic	<16 bits	$O\left(N^{1/3}\right)$ [8]

5.2 Sub-Exponential Algorithms Summary

Table 2. Summary of Sub-Exponential-Time Factorization Algorithms

Algorithm	**Basis**	**Effective Range**	**Complexity**
Dixon	Random congruences	<73 bits	$L(1/2, c)$ [9]
CFRAC	Continued fractions	<80 bits	$L(1/2, c)$ [10]
Quadratic Sieve	Sieving	<100 digits	$L(1/2,1)$ [11]
ECM	Elliptic curves	<83-bit factors	$L(1/2, \sqrt{2})$ [12]
NFS	Number fields	>110 bits	$L\left(1/3, (64/9)^{1/3}\right)$ [13]

5.3 Practical Observations

Empirical results from the original dataset indicate that exponential algorithms degrade rapidly as the bit length increases, whereas sub-exponential methods continue to perform reliably. NFS overwhelmingly outperforms all other classical algorithms on

large inputs, confirming its status as the practical tool for analyzing RSA-size integers [2].

6. Conclusion

The analysis presented throughout this paper highlights how integer factorization has gradually evolved from simple, intuitive methods to highly sophisticated algorithms capable of challenging even large cryptographic structures. Exponential algorithms - such as Fermat's method, Pollard's ρ, SQUFOF, Pollard's p−1, and others-demonstrate how early approaches often relied on elegant mathematical observations yet struggled as the numbers grew larger. Their limited scalability makes them more suitable for teaching, experimentation, or factoring integers with very specific structural properties.

Sub-exponential algorithms mark a turning point. Dixon's method and CFRAC introduced new ways of thinking about smooth numbers and congruences. The Quadratic Sieve refined these ideas into a practical, well - engineered method that remained dominant for many years. Lenstra's Elliptic Curve Method added an entirely new dimension by showing that elliptic curves, originally studied for purely theoretical reasons, could offer remarkable advantages in finding small and medium-sized factors. Finally, the Number Field Sieve stands as the culmination of decades of progress - a complex and powerful algorithm that currently defines the limits of classical factorization.

Across all these methods, a consistent theme emerges: the difficulty of factoring large integers is not a fixed barrier but a moving target. As algorithms improve and as computational resources grow, the effective security of cryptographic systems must be reassessed. Although modern RSA implementations remain secure when appropriate key sizes are used, history shows that relying on static assumptions can be dangerous. Each new algorithm reshapes the landscape, sometimes subtly, sometimes significantly.

In a broader sense, this comparison reminds us that cryptographic security is intimately tied to mathematics, and mathematics itself is always evolving. Understanding the strengths and weaknesses of both exponential and sub - exponential

factorization algorithms is therefore not merely an academic exercise - it is an essential part of anticipating future risks, evaluating current systems, and guiding the transition toward more resilient cryptographic frameworks.

Ultimately, integer factorization remains both a practical challenge and a fascinating mathematical journey. As long as cryptography depends on the hardness of factoring, studying these algorithms-and the ideas behind them - will continue to be a vital part of ensuring secure communication in the digital world.

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