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Gaussian-based variational approximation of one-dimensional quantum droplets

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On the other hand, large colliding droplets may merge or suffer fragmentation, depending on their relative velocity. The frequency of a breathing excited state of droplets, as predicted by the dynamical variational approximation based on the Gaussian ansatz, is found to be in good agreement with numerical results. Finally, the stability diagram for a single droplet with respect to shape excitations with a given wavenumber is drawn, being consistent with preservation of the Weber number for large droplets.

Keywords: Gaussian *ansatze*, quantum droplets, variational approximation

I. Introduction

The equilibrium densities of both components of the binary condensate depend on interaction strengths and atomic masses of the two species. The symmetric case of equal masses and strengths of intraspecies interactions, with equal numbers of atoms in each components, allows a simpler and more elegant description. In this case, the density profiles of both components coincide and can be described by the effective three-dimensional (3D) single-component GPE with cubic and quartic nonlinearities, the latter term representing the LHY correction [2].

This possibility provides an important interdisciplinary connection to the field of nonlinear optics [4], as concerns the underlying model equations with higher-order nonlinearities [5–7] and, possibility, controlled generation of solitons in these systems. On the other hand, in the case when two-component features in the dynamics are essential, they may be affected by an additional linear interconversion between the components [8].

In three and two dimensions, quasi-1D solitons are unstable with respect to the transverse snake instability, although the stability can be enhanced by imposing rotation to the quantum droplets [9]. The advantage of the proper 1D geometry, imposed by the tight confinement in the transverse directions (cf. the experimental realization of the

Tonks-Girardeau gas [10, 11]), is that such an instability is absent, thus permitting one to realize a very clean and highly controllable many-body testbed which may permit the measurement of quantum many-body effects with very high precision.

Commonly known hallmarks of solitons are being (i) self-trapped and (ii) robust with respect to soliton-soliton collisions. While the former feature is definitely present in quantum droplets, the latter one should be yet verified. It was proposed to use Gaussian *ans¨atze* for gaining an analytical insight in physics of dipolar [3] and BEC [1] droplets. In particular, the dynamical version of the Gaussian-based variational approximation (VA) can be used to predict the frequency of intrinsic oscillations of the soliton-like objects in an excited state [12-14]. Excitations in a dipolar quantum droplet have been experimentally studied and a scissors mode has been observed in it [15]. Recently, intrinsic modes were theoretically investigated in Fermi-Bose mixtures [16], spin-orbit-coupled Bose-Einstein condensates (BECs) [17], including those dominated by the LHY terms [18], and in a discrete BEC model [19].

The collisions between dipolar droplets were experimentally studied in Ref. [20]. The system was confined to an elongated trap and the interactions were quenched in such a way that the density distribution was split into multiple pieces. The droplets which have been formed were shown to be long lived and their dynamics was studied. Until now no similar experiment has been performed with short-range interacting droplets although such experimental studies might be expected in the near future. The goal of our study is to analyze the dynamic properties of ultradilute quantum droplets.

In the present work we address collisions of 1D quantum droplets and intrinsic oscillations of an isolated droplet, along with excitations generated by imprinting onto it a density modulation with a certain wavenumber. The article is organized as follows. First, we address static droplets in Sec. II addresses the asymptotic analysis in the limits of small and large droplets. In subsection II1 we develop the Gaussian *ansatz* for the study of both stationary and dynamical properties of the droplet. The conditions of applicabil- ity of the Gross-Pitaevskii equation for describing statical and dynamical properties are analyzed in Sec. II.

II. Application conditions of variational approximation of quantum drops

In this subsection we approximate the shape of the droplet by a Gaussian and optimize its width according to the variational principle. This simple model provides additional insight in properties of the droplets as the number of particles varies.

The underlying time-dependent GPE for the onedimensional droplet with the symmetric components is [2]

$$i\hbar\psi_t = -\frac{\hbar^2}{2m}\psi_{xx} + \delta g|\psi|^2\psi - \frac{\sqrt{2m}}{\pi\hbar}g^{\frac{3}{2}}|\psi|\psi\,, \qquad (1)$$

casts Eq. (1) in an equation without free coefficients (where the primes are omitted):

$$i\psi_t + \frac{1}{2}\psi_{xx} - |\psi|^2\psi + |\psi|\psi = 0\,, \qquad (2)$$

A peculiarity of the 1D geometry is that the groundstate solution of the GPE for the droplet, Eq. (2), can be found in an explicit form [9]:

$$\psi_{exact}(x) = -\frac{3\mu\exp(-i\mu t)}{1+\sqrt{1+\frac{9\mu}{2}}\cosh\left(\sqrt{-2\mu x^2}\right)}\,, \qquad (3)$$

$$\mu = -\frac{1}{2^{\frac{1}{3}}3^{\frac{2}{3}}}N^{\frac{2}{3}} = -0.382N^{\frac{2}{3}}\,, \qquad (4)$$

$$\sqrt{\langle x^2\rangle} = \frac{L}{2\sqrt{3}} = \frac{N}{2\sqrt{3}n_0} = 0.65\,. \qquad (5)$$

The VA is based on the Lagrangian for Eq. (3),

$$L = \int_{-\infty}^{+\infty}\mathcal{L}dx\ , \qquad (6)$$

$$\mathcal{L} = \frac{i}{2}(\psi\psi_t^* - \psi^*\psi_t) + \frac{1}{2}|\psi_x|^2 + \frac{1}{2}|\psi|^4 - \frac{2}{3}|\psi|^3\,. \qquad (7)$$

To develop the dynamical version of the VA, we adopt the Gaussian *ansatz*,

$$\psi = A(t)\exp\left[i\phi(t) - \frac{x^2}{2\left(W(t)\right)^2} + ib(t)x^2\right], \qquad (8)$$

where A, ϕ, W, and b are real amplitude, phase, width and chirp, respectively (in the time-independent version of the VA, $b = 0$). Although the long-distance Gaussian asymptotic form of wave function (8) is incompatible with the exponential decay of the exact solution (3), we demonstrate below that the overall accuracy provided by the VA is extremely good. The normalization of the wave function determines the number of particles in the droplet.

$$N = \int_{-\infty}^{+\infty}|\psi(x)|^2dx = \sqrt{\pi}A^2W\,. \qquad (9)$$

Substituting the *ansatz* in Lagrangian density (7) and using Eq. (9) to eliminate A^2 in favor of N, as $A^2 = N/\pi^{1/2}W$, one can produce the effective Lagrangian:

$$\frac{L_{VA}}{N} = \frac{d\phi}{dt} + \frac{W^2}{2}\frac{db}{dt} + \frac{1}{4W^2} + W^2b^2 + \frac{N}{2\sqrt{2\pi}W} - \frac{2^{3/2}\sqrt{N}}{3^{3/2}\pi^{1/4}\sqrt{W}} \qquad (10)$$

The Euler-Lagrange equations for variables b and W are derived from here:

$$b = \frac{1}{2W}\frac{dW}{dt}\,, \qquad (11)$$

$$\frac{d^2W}{dt^2} = \frac{1}{W^3} + \sqrt{\frac{1}{2\pi}}\frac{N}{W^2} - \frac{2^{\frac{3}{2}}N^{\frac{1}{2}}}{\pi^{\frac{1}{4}}(3W)^{\frac{3}{2}}} \equiv -\frac{dU_{eff}}{dW}\,, \qquad (12)$$

where the effective potential for oscillations of the soliton's width is

$$U_{eff}(W) = \frac{1}{2W^2} + \sqrt{\frac{1}{2\pi}\frac{N}{W}} - \frac{2}{\pi^{\frac{1}{4}}}\left(\frac{2}{3}\right)^{\frac{3}{2}}\sqrt{\frac{N}{W}}\,. \qquad (13)$$

This potential gives rise to a shallow well, as shown in *Fig. 1*

In the framework of the VA, the stationary soliton corresponds to the minimum of potential (11). Its width is determined by the cubic equation for $\sqrt{W}$, which follows from condition for the potential minimum, $dU_{\mathrm{eff}}/dW = 0$, as per Eq. (12):

$$1+\sqrt{\frac{1}{2\pi}}NW-\frac{1}{\pi^{\frac{1}{4}}}\left(\frac{2}{3}\right)^{\frac{3}{2}}\sqrt{N}W^{\frac{3}{2}}=0\,.\quad(14)$$

In particular, in both asymptotic limits of $N\to 0$ and $N\to\infty$ the width is large:

$$W(N\to 0)\approx\frac{3}{2}\pi^{\frac{1}{6}}N^{-\frac{1}{3}},\qquad(15)$$

$$W(N\to\infty)\approx\frac{27}{16\sqrt{\pi}}N\,.\qquad(16)$$

It is also possible to find the exact *minimum value* of the VA-predicted width,

$$W_{min}=\frac{3\sqrt{3}}{2^{3/4}}\approx 3.09\qquad(17)$$

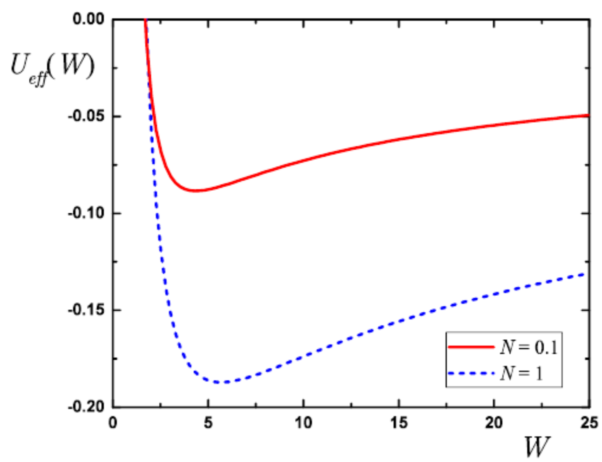

Figure 1: (Color online) The plot of the effective potential (13) for N = 0.1 and N = 1. The minimum of the potential defines the optimal width W of the droplet.

which is attained at the value of the norm of the order of one, namely

$$N(W_{min})=\frac{2^{5/4}}{3^{3/2}}\approx 0.81\;.\qquad(18)$$

The mean-square size of the droplet with the Gaussian profile is $\langle x^2\rangle=W^2(N)/2$. In the $N\to 0$ limit, the width is given by Eq. (15), the respective mean-square size increasing as $N^{-1/3}$:

$$\sqrt{\langle x^2\rangle}=\frac{3}{2\sqrt{2}}\pi^{1/6}N^{-1/3}\approx 1.28N^{-\frac{1}{3}}.\qquad(19)$$

Comparison with Eq. (6) makes it evident that the droplet has a minimum size at $N\sim 1$.

The frequencies of low-lying collective excitations are determined by the eigenvalues of the Hessian matrix evaluated at the equilibrium position [13]. In the 1D system, it reduces to the second derivative

$$\frac{d^2 U_{eff}(W_{min})}{dx^2} = \frac{1}{2}\omega^2 , \qquad (20)$$

and corresponds to the "breathing" or compressiondilatation mode of frequency ω. For limit values of N the frequency takes the approximate form,

$$\omega(N \to 0) = \frac{2\sqrt{\frac{2}{3}}}{3\sqrt[3]{\pi}} N^{2/3} , \qquad (21)$$

$$\omega(N \to \infty) = \frac{322^{1/4}\sqrt{\frac{\pi}{3}}}{81}\frac{1}{N} .(22)$$

The ground-state energy is obtained by evaluating the energy functional

$$E = \int_{-\infty}^{+\infty} \left(\frac{1}{2}|\psi_x|^2 + \frac{1}{2}|\psi|^4 - \frac{2}{3}|\psi|^3\right) dx , \qquad (23)$$

and the chemical potential is then obtained as its derivative with respect to the number of particles, $\mu = dE/dN$. The chemical potential of a droplet is negative and approaches zero, in the limit of $N \to 0$, as

$$\mu = -\frac{5N^{\frac{2}{3}}}{9\pi^{\frac{1}{3}}} = -0.379 N^{\frac{2}{3}} .(24)$$

Note that the difference of this VA-produced result with the exact one, given by Eq. (4), is less than 1%.

An important issue is to clarify the regions of applicability of the GPE (1) for describing static and dynamic properties of the droplets. The GPE has been proven to be immensely useful for description of experiments with single-component ultracold Bose gases, as the mean-field description it provides is sufficient for interpretation of a large variety of quantum effects [21]. On the other hand, artificial incorporation of terms with higher powers of the condensate wave function in the GPE in order to "simulate" LHY terms in the energy may lead to incorrect dynamics of excitations in the framework of the amended equation.

III. Conclusion

The key point here is that the dominant contribution to the BMF energy come from distances smaller than or comparable to the healing length ξ, defined, in terms of the sound velocity, c, by $\hbar^2/(2m\xi^2) = mc^2$. In order to treat the BMF term as a "local" contribution, the distances at which the density profile changes should be large compared to the healing length. In a singlecomponent gas this is impossible, as the density profile changes exactly at distances $\sim \xi$. In quantum droplets, the situation is completely different, as the variation in the density profile is governed by the "soft" healing length

$$\xi_- = \frac{\hbar}{\sqrt{2}mc_-} ,$$

while the BMF energy is earned at the distances comparable to the "hard" healing length,

$$\xi_+ = \frac{\hbar}{\sqrt{2}mc_+}.$$

The speed of sound of the two modes is defined by $mc^2_\pm = (g \pm |g_{\uparrow\downarrow}|)n$ in the symmetric case [2]. Thus, the coupling constant δg defines the speed in the soft mode, $mc_-^2 = (\delta g)n$, while $g \gg \delta g$ produces a much larger speed in the hard mode, $mc_+^2 \approx gn$.

The large separation of scales, $\xi_- \gg \xi_+$, justifies the inclusion of higher-order terms in the GPE as local ones. Accordingly, the relatively slow dynamics, which takes place at timescales large compared to the typical "hard" time interval,

$$t_+ = \frac{2m\xi_+^2}{\hbar}$$

is correctly described by the amended GPE (1).

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